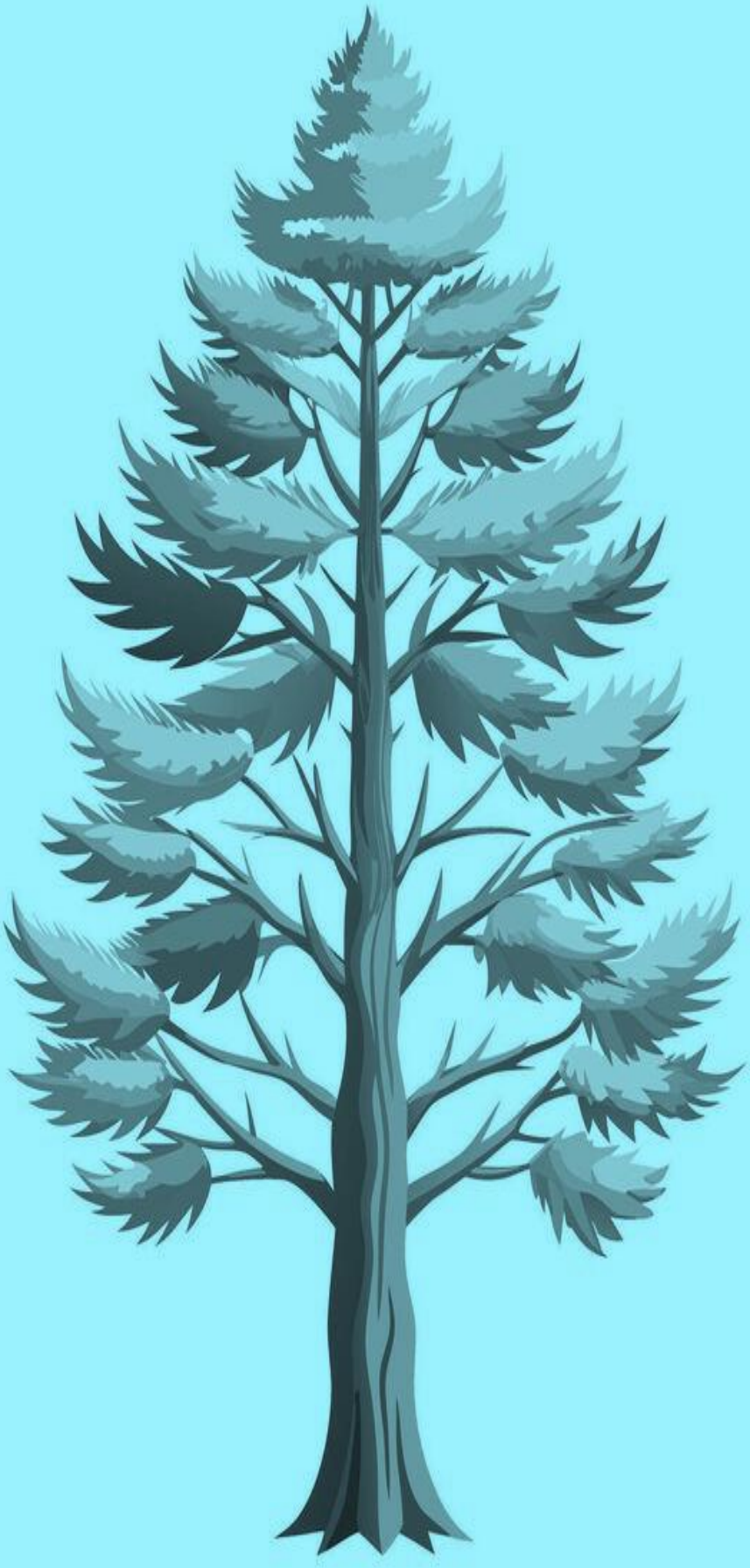

Linear Algebra

notions and problems

In progress…

Mohammed Fulano

B.Sc. in Applied Mathematics

Algeria

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mohammedfulano@hotmail.com

Linear Algebra is an subject that is as old as mathematics. Greek mathematicians used a sort of linear proprieties in their investigations in geometry. Unlike many mathematical disciplines, Linear Algebra has become an independent subject only quite recently. It is today an essential prerequisite not only for studying geometry but for many other branches of science like: physics, economics, functional analysis, …. So the scientific student has to master its methods.

Vectors:.....3
Linear systems :.....3
Matrices:.....3
Vector spaces:.....3
Linear maps:.....3
Spectrum and eigenvectors:.....3
Reduction.....3

Vectors

A vector is an *ordered* collection of numbers. It can be either of the forms

$$(a_1, \dots, a_n) \quad , \quad \|a_1, \dots, a_n\| \quad , \quad [a_1, \dots, a_n]$$

or of the forms

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad , \quad \left\| \begin{matrix} a_1 \\ \vdots \\ a_n \end{matrix} \right\| \quad , \quad \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Each one of the first type is called a *row* vector[1] , the second type is called a *column* vector. A vector of n component is called also an *n-tuple*. There is no difference between the two. Here we use the notation

$$(a_1, \dots, a_n) \quad \text{and} \quad \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

A *vector* comes from geometry in the usual plane $\mathbb{R}^2$, and usual space $\mathbb{R}^3$. Afterwards the word took other general significations. So in modern, we can mean by a vector a function.

[1] The word vector was invented by physicists in the $XVIII$ century then by Irish William **Hamilton**$(1805 - 1865)$.

The natural example of a vector is the coordinates of a point in the space $\mathbb{R}^n$. Let p a point in $\mathbb{R}^n$, then it has the *coordinates* $(a_1, \dots, a_n)$. If we take $n = 2$ we find the coordinates of a point p in the plane the following is the vector or the point $(2,3)$

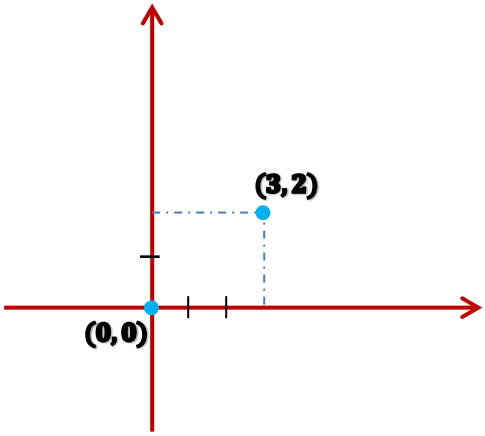

Fig. a vector(point) in the plan $\mathbb{R}^2$.

Let $v_1, \dots, v_n$ n vectors, a *linear combination* of $(v_1, \dots, v_n)$ is another vector of the form

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

where $a_1, \dots, a_n$ are real numbers, in order to distinguish them from vectors there are called *scalars*. In these notes, scalars can be either real or complex numbers.

We note the *null vector* by $\mathbf{0_n}$ which is by definition

$$\mathbf{0_n} = (0, \dots, 0) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

The zero n −tuple . We note some properties of the algebra of vectors[2]

$$v + (-v) = \mathbf{0_n}$$
$$v_1 + v_2 = v_2 + v_1$$
$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$$

Scalar product of two vectors :

[2] Note that for numbers and the law " − " : $2 - 3 \neq 3 - 2$, and $2 - (1 - 2) \neq (2 - 1) - 2$.

Let $v_1(a_1, \dots, a_n)$ and $v_2(b_1, \dots, b_n)$ two vectors, we define the *scalar product* of v_1 and v_2 as the product

$$(v_1, v_2) = \sum_{i=1}^{n} a_i b_i$$

another notation is $v_1 \circ v_2$. If $(v_1, v_2) = 0$, we say that v_1 and v_2 are *orthogonal* or perpendicular. Another relation verified by the scalar product is :

$$v_1 \circ v_2 = |v_1||v_2| \cos(\alpha)$$

The positive $|v_1 \circ v_2| = |v_1||v_2||\cos(\alpha)|$. We notice that if $\alpha = \frac{\pi}{2}$ then $v_1 \circ v_2 = 0$ as expected. The dot product $(x, y) = \sum_1^n x_i y_i$ is a special case of what we call a *bilinear form* . The scalar product is a *symmetric* bilinear form. We have this fundamental inequality

$$|v_2 \circ v_1| \leq |v_2||v_1|$$

Called Cauchy–Schwarz inequality[3], if $v_2 = kv_1$ then $|v_2 \circ v_1| = |v_2||v_1|$.

Norm of a vector:

A vector v as a point on the line $\mathbb{R}$ has a distance a from the point $\mathbf{0_n}$ which is positive lenght

$$|v| = a \geq 0$$

in the usual plane $\mathbb{R}^2$, the length from v to the origin $\mathbf{0_n} = (0,0)$ is

$$|v| = \sqrt{{a_1}^2 + {a_2}^2}$$

in the usual space $\mathbb{R}^3$, the length to the origin $\mathbf{0_n} = (0,0,0)$ is

$$|v| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$$

[3] Due to French Augustin **Cauchy**$(1789 - 1857)$ and German Hermann **Schwarz**$(1843 - 1921)$ and also Victor **Bunyakovsky**$(1804 - 1889)$ so it is also known under the name the **Cauchy–Schwarz–Bunyakovsky** inequality. it is a special case of the more general inequality due to German Otto **Hölder**$(1859 - 1937)$: $\sum |a_i b_i| \leq \left(\sum |a_i|^p\right)^{1/p} \left(\sum |b_i|^q\right)^{1/q}$ where $\frac{1}{p} + \frac{1}{q} = 1$.

The above defined length has some proprieties like

i) $|x+y| \leq |x| + |y|$
ii) $|x-y| \leq |x| + |y|$

There is no problem to generalize the length to $\mathbb{R}^n$ where $n \geq 4$ ans write

$$\|v\| = \sqrt{{a_1}^2 + \cdots + {a_n}^2}$$

In this case the length becomes the *norm* of v and noted by the symbol $\| \ \|$. For $\mathbb{R}$ we find $\| \ \|$ is a symbol for $| \ |$

$$\|v\| = \sqrt{{a_1}^2} = |a|$$

Mathematicians have invented[4] many norms $\| \ \|$, however to be in the place of a length, the norm $\| \ \|$ has to verify three conditions:

Definition

$i)$ $\|x\| \geq 0$
$ii)$ $\|x\| = 0 \Leftrightarrow \|x\| = 0$
$iii)$ $\|\lambda x\| = |\lambda| \|x\|$
$iv)$ $\|x+y\| \leq \|x\| + \|y\|$

we note that the usual length $| \ |$ verifies the above conditions. The norm has some proprieties

i) $\|x+y\| \leq \|x\| + \|y\|$
ii) $\|x-y\| \leq \|x\| + \|y\|$

Linear independence

A collection of vectors $v_1, \ldots, v_n$ are *linearly independent* if the equality

[4] Like $\|v\|_p = (\sum_1^n |x_i|^p)^{1/p}$ and $\|v\|_\infty = \max_i |x_i|$

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$$

implies that all $a_1 = a_1 = \cdots = 0$. To illustrate it simply, we say that two vectors v_1 and $v_2$are independent if we cannot express a vector in terms of the other v_2. The *span* of a vector v is the that all vectors of the form $\lambda\, v$ with $\lambda \in \mathbb{R}$. The span of the vector $(1,0)$ in $\mathbb{R}$ is the line $x(1,0)$ or $(x, 0)$ that is the line real axis. In many occasions we ask if a collection of vectors can span a space.

Theorem

A set of n linearly independent vectors span the space $\mathbb{R}^n$.

Examples:

–The vector $\{(1,0)\}$ can span $\mathbb{R}$. Any vector $\{(a, 0)\}$ can span $\mathbb{R}$.
–The vectors $\{(1,0), (0,1)\}$ can span all the plan $\mathbb{R}^2$. Any point in the plan is of the form (x, y) can be as $(x, y) = x(1,0) + y(0,1)$.

⏏

Vectors in physics

The notion of vector exists in physics too, for an example a force $\vec{\mathrm{F}}$ is a vector[5]. A vector in physic

$$i + j + k$$

In mathematics we make no difference between a vector and a point but in physics the situation is different: a vector is different from a point. A physical vector has :

–a direction.
–a sense.
–a magnitude(length).

For a vector in physics the magnitude is $\left\|\vec{\mathrm{F}}\right\|$ in many occasions we can have $\left\|\overrightarrow{\mathrm{F}_1}\right\| = \left\|\overrightarrow{\mathrm{F}_2}\right\|$ although $\overrightarrow{\mathrm{F}_1} \neq \overrightarrow{\mathrm{F}_2}$. The difference is that mathematical vectors are abstract entities. Two vectors v_1, v_2, have the same direction if $v_1 = kv_2$ where k is positive.

Vectors in geometry

[5] The word vector already Introduced in astronomy in the $XVIII$ century. It comes from Latin "*vehere*" meaning *carrier*. In 1846, introduced **Hamilton** modern meaning vector and scalars.

So here we find a strong link between geometry and linear algebra.to scalar product we have the *vector* product

$$v_1 \times v_2 = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Two vectors v_1, v_2 are parallel if $v_1 \times v_2 = 0$. The magnitude of the vector product $v_1 \times v_2$ presents the area by the two vectors and its value given by a formula similar to dot product

$$|v_1 \times v_2| = |v_1| \bullet |v_2| \bullet |sin(\alpha)|$$

In geometry , vector function comes from the motion of a particle in mechanics. Two point A, B we form the vector $A\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, $B\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ with the coordinates $\overrightarrow{AB}\begin{pmatrix} b_1 - a_1 \\ \vdots \\ b_n - a_n \end{pmatrix}$ so any vector is equivalent to a vector with the beginning at the origin 0_n. The same is true for . Physical vectors in $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$ are vectors of geometry. For example they are subject to the *parallelogram law* of addition.

Exercices

♦ is $(0,0)$ a linear combination of $(1,-1)$ and $(2,3)$?
♦ is $(1,1)$ a linear combination of $(1,-1)$ and $(2,3)$?
♦ Show that $v = \mathbf{0}$ is free.
♦ Show that if v is not free then $v = \mathbf{0}$.
♦ Show that if we add $v = \mathbf{0}$ to any collection of vector we a dependent of vectors.
♦ Show that two vectors are parallel(collinear) iff they are dependent.
♦ show that if v_1, v_2, v_3, are free then the same with $(v_1 + v_2, v_2 + v_3, v_1 + v_3)$.
♦ Let $v_1, \dots, v_n$ are free vectors, show that $v_1, \dots, v_k$ $(n > k)$ are free.
♦ Let v_1, v_2 two vectors in $\mathbb{R}^n$. Show that:
– $|v_1 + v_2| \leq |v_1| + |v_2|$
– $|v_1 + \cdots + v_n| \leq |v_1| + \cdots + |v_n|$
– $|v_1 - v_2| \leq |v_1| + |v_2|$
– $|v_1 - \cdots - v_n| \leq |v_1| + \cdots + |v_n|$
– $|v_1 - v_2| \geq |v_1| - |v_2|$.
♦ show that the Cauchy–Schwarz inequality is a special case of Hölder inequality.
♦ show that $\|1\| = 1$
♦ show that $v \circ v = |v|^2$
♦ show that $v_1 \circ v_2 = v_2 \circ v_1$
♦ show that $(v_1 + v_2) \circ (v_3 + v_4) = v_1 \circ v_3 + v_1 \circ v_4 + v_2 \circ v_3 + v_2 \circ v_4$

♦ find the unit vector v_n in the direction of the vector $\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$.

♦ show that unit vector v_u has the norm $\|v_u\| = 1$.

♦ show that $\big\|\|v\|\big\| = \|v\|$.

♦ show that if $\|\alpha x\| = |\alpha|\|x\|$ then : $x = 0 \Longrightarrow \|x\| = 0$.

♦ show second triangular inequality :$\|x - y\| \geq |\|x\| - \|y\||$

Linear systems

A *linear equation* is any of the form

$$a_1x_1 + a_1x_2 + \cdots + a_nx_n = b$$

where the exponents of every unknown $x_1, \ldots, x_n$ is 1. A *linear system* is composed of one or more linear equations. Linear systems come from many applications in science. Of course the main problem is how to find the solution of the system. A general linear system is of the form

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{gathered}$$

It is composed of m equations in n *unknowns* which is a vector $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. The simplest linear system is composed of one linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

which may have in general infinitely many solution if $a_k \neq 0$. For example the equation

$$-x_1 + 3x_2 + x_3 = -2$$

has infinitely many solutions and $-4x_1 = 3$ has one solution. The equation

$$0x_1 + 0x_2 + \cdots + 0x_n = b$$

is a *degenerate* equation. It has no solution if $b \neq 0$ and infinitely many solutions if $b = 0$.

For a general linear system, we have three possibilities : $n = m$, $n > m$ or $n < m$.

$-n = m$ (the number of equations is equal to the number of unknowns)

$$\begin{aligned} x - y + t &= 1 \\ x + 2y + t &= 2 \\ x - y + t &= 1 \end{aligned}$$

$-n < m$ (the number of equations is bigger than the number of unknowns)

let

$$\begin{aligned} x - y &= 1 \\ x + 2y &= 1 \\ x - y &= 1 \end{aligned}$$

$-n > m$ (the number of unknowns is bigger than the number of equations) there infinity many solutions of such system

$$\begin{aligned} x - y + z &= 1 \\ 2y - 2z &= 1 \end{aligned}$$

A simple linear system(which is rare and easy to solve) is the *triangular* system, it belongs to the type $n = m$ (We will see when a linear system has a solution or not).

Let us take an example when $n = 3$:

$$\begin{aligned} x + 2y - z &= 1 \\ 3y + z &= 2 \\ -z &= 1 \end{aligned}$$

We find the solution (x, y, z) by back substitution $v = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$. If we look for a generalization of triangular system we find the *echelon* form.

The echelon form

the following linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\

Let us name the coefficient a_{11} *the pivot* the all under are 0

An important linear system is the following

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \ldots\ldots\ldots\

Matrices

The origin of matrices is linear systems. Mathematicians wrote a linear system when $n = m$ like this one

$$ax + by = c$$
$$dx + ey = f$$

in a compact form

$$\begin{pmatrix} a & b \\ d & e \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ f \end{pmatrix}$$

where $\begin{pmatrix} a & b \\ d & e \end{pmatrix}$ the coefficients is called a *matrix*[7] . The precedent matrix has two rows $\{(a, b), (b, e)\}$ and columns $\left\{\begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} b \\ e \end{pmatrix}\right\}$. It has two rows and 2 columns then it is a 2×2 matrix. a matrix can be $n \times m$, not necessarily $n = m$.

there are three possible notation for a matrix

$$(a_{ij}) \quad , \quad \|a_{ij}\| \quad , \quad [a_{ij}]$$

The matrix its coefficients a_{ij} all 0 is the *null matrix*

$$\mathbf{0_{nm}} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

We note the null matrix of order n by the symbol $\mathbf{0_{nn}}$. The set of all matrices is noted

[7] The Latin word matrix was introduced by English James **Sylvester**$(1814 - 1897)$ friend of **Cayley** in 1850. The word *matrix* means *womb*.

$\mathcal{M}_{mn}(K)$ where K is the set of the coefficients can be $\mathbb{R}$ or $\mathbb{C}$ for example. We add to so for a two linear system

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Or simply if we put $\begin{pmatrix} a_{11} & a_{11} \\ a_{11} & a_{11} \end{pmatrix} = A$, $X = \begin{pmatrix} x \\ y \end{pmatrix} = X$ and $\begin{pmatrix} a \\ b \end{pmatrix} = B$, the linear system is

$$AX = B$$

So instead we all the linear system we are led to study the matrix $\begin{pmatrix} a_{11} & a_{11} \\ a_{11} & a_{11} \end{pmatrix}$ itself.

Using the addition between (n, m) matrices (it is possible that $n = m$) , matrices we observe that:

Theorem

$-A + B = C$
$-(A + B) + C = A + (B + C)$
$-A + (-A) = \mathbf{0_{nm}}$

Product between matrices was a new operation in XIX century

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}(b_1, \dots, b_n) = \begin{pmatrix} a_1 b_1, \dots, a_n b_n \\ \dots \\ a_n b_1, \dots, a_n b_n \end{pmatrix}$$

But

$$(a_1, \dots, a_n)\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n$$

So the rule is

$$(n, p)(p, m) = (n, m)$$

a is I_n

$$I_n = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

The all diagonal elements is 1 and all other are 0. $\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} = \delta_{ij}$ where is the *Kronecker symbol* as

$$\delta_{ij} = \begin{cases} 1 \ if \ i = j \\ 0 \ if \ i \neq j \end{cases}$$

Need not be squared : $IA = AI$.

Like

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

But

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

the product of two matrices A and B is commutative which means that $AB \neq BA$ in general but we have :

Theorem

$-A(BC) = (AB)C$

A matrix can formed of complex numbers like the 2×2 matrix $\begin{pmatrix} i & 2i \\ 1+i & -i \end{pmatrix}$. In this new a vector is matrix with one.

A *conjugate* of a matrix is $\overline{\mathrm{A}}$. for $A = \begin{pmatrix} i & 2i \\ 1+i & -i \end{pmatrix}$, the conjugate is

$$\overline{\begin{pmatrix} \imath & 2\imath \\ 1+\imath & -\imath \end{pmatrix}} = \begin{pmatrix} \overline{\imath} & \overline{2\imath} \\ \overline{1+\imath} & \overline{-\imath} \end{pmatrix} = \begin{pmatrix} -i & -2i \\ 1-i & i \end{pmatrix}$$

If all the coefficients are real then : $\overline{\mathrm{A}} = A$.

Remarks:

–nanother of the product of matrices is that the equation $AB = \mathbf{0_{nn}}$ does not imply either $A = \mathbf{0_{nn}}$ or $B = \mathbf{0_{nn}}$. A *nilpotent* matrix A is such has the propriety $A^k = 0_{nn}$ for some positive integer k⏏

A matrix is diagonal if $a_{ij} = 0$ for $i \neq j$. So the concept of *trace* (of course of a squared matrix) which is simply the sum of the diagonal coefficients $Tr(A) = \sum_{i=1}^{n} a_{ii}$. Some proprieties of the trace:

Proposition:

$-Tr(AB) = Tr(BA)$
$- Tr(A + B) = Tr(A) + Tr(B)$

The trace of a matrix can be found in formulas like in differential geometry. A matrix can have different interesting forms. The *lower* triangular matrix

$$\begin{pmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

And the *upper* triangular

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix}$$

Symmetric matrices:

if is $v = (a_1, \dots, a_n)$ its transpose is the vector

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

which we note v^T. The scalar product $v_1 \circ v_2$ is just $v_1 \circ v_2 = v_1 {v_2}^T$. A *transpose* of a matrix is for the same the become the columns s and the columns becomes the rows. Let (a_{ij}) be a matrix, then

$$(a_{ij})^T = (a_{ji})$$

Some proprieties

Proposition:

$-(A + B)^T = A^T + B^T$
$-(AB)^T = B^T A^T$

An important matrix is the symmetric matrix

Definition

A squared matrix A is *symmetric* if $A^T = A$.

The unit matrix I is symmetric. A matrix is called *self-adjoint* or Hermitian[8] if $\overline{A^T} = A$. Note that $\overline{A^T} = (\overline{A})^T$. A Hermitian matrix is noted A^*, so we have

$$A^* = \overline{A^T} = A$$

Every real symmetric is Hermitian . we have the propriety

$$(AB)^* = A^* B^*$$

A *normal* matrix is a matrix verifying $AA^* = A^* A$.

A matrix is *skew* if $A^T = -A$.

Determinant of a Matrix:

Let see from where came the idea of a *determinant*. Let an example when $n = r$. a simple linear system is of two equations

$$ax + by = c$$
$$dx + ey = f$$

the solution is a vector $\begin{pmatrix} x \\ y \end{pmatrix}$. If by substitution we find

$$x = \frac{bf - ec}{ae - bd} \quad , \quad y = \frac{af - cd}{ae - bd}$$

[8] After French mathematician **Hermite**$(18 - 1)$.

The number $\boldsymbol{ae - bd}$ is called *determinant*[9] , an important factor in linear systems studied later. we can remark at once that if $ae - bd = 0$ there is a problem in our system (namely there could be no solutions).

We introduce *determinants* which have an independent theory[10]. A determinant is a real number that has with the solution of linear systems. For example the determinant[11] of the matrix $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 1 \bullet 1 - 2 \bullet 2 = -3$$

For general, it is by the alternated *Laplace expansion*

$$\sum_{i=1}^{n} (-1)^{i+j} a \quad = a \quad - a \quad + a \quad - \cdots$$

Let $A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, the determinant of A is

$$det(A) = 1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

For $n = 3$ there is an extra different method called *Sarrus method* , due to French mathematician Pierre **Sarrus**$(1798 - 1861)$. For the matrix $\begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ so the determinant

$$\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \begin{matrix} 1 & 3 \\ 2 & 1 \\ 1 & 1 \end{matrix} = (1 \bullet 1 \bullet 1 + 3 \bullet 1 \bullet 1 + 1 \bullet 2 \bullet 1) - (3 \bullet 2 \bullet 1 + 1 \bullet 1 \bullet 1 + 1 \bullet 1 \bullet 1) = -2$$

the determinant of a matrix has the properties

Proposition:

$-det(AB) = det(A)det(B)$

[9] Used by many mathematicians, the name coined by **Gauss**.
[10] See *The Theory of Determinants in the Historical Order of Development* of Thomas **Muir**$(18 - 1)$.
[11] The symbol two vertical | |was introduced by **Cayley** in 1841.

$-det(AB) = det(BA)$
$-$ If $\alpha \in \mathbb{R}$ then : $det(\alpha B) = \alpha^n det(A)$.
$-det(A^T) = det(A)$

Remarks:

$-$Keep in mind that a non-squared matrix has no determinant.
$-$The determinant of diagonal, triangular, L and , U matrices is just $\prod_1^n a_{ii}$ ⏏

Inverse of a Matrix:

To the idea of inverse of a matrix let is linear system $n = 2$. In (y_1, y_2)in terms of (x_1, x_2) in

$$y_1 = ax_1 + bx_2$$
$$y_2 = cx_1 + dx_2$$

We inverse the (x_1, x_2) in terms of (y_1, y_2)

$$x_1 = ey_1 + fy_2$$
$$x_2 = gy_1 + hy_2$$

It is not always possible to do so. Let A be a square matrix, we define the inverse as A as a matrix A^{-1} such that $A^{-1}A = I$ we have

$$A^{-1}A = AA^{-1} = I$$

we say that is *invertible*. Note that if A is not square , it has no inverse. The inverse of A is unique. If A has an inverse it is a regular matrix, otherwise it is *singular*. The matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is singular.

Proposition :

$-(AB)^{-1} = B^{-1}A^{-1}$
$-(A^{-1})^T = (A^T)^{-1}$
$-(A^{-1})^* = (A^*)^{-1}$

A matrix A of n linearly independent rows (or columns) is invertible. When $A^{-1} = A^T$, A is

orthogonal, we have then $AA^T = I$.

Definition

An *unitary* matrix is a matrices such that : $A^{-1} = A^*$.

Matrices in linear systems:

A matrix can a have an inverse noted A^{-1}. the idea comes from a linear system take the simple

$$ax + by = c$$
$$dx + ey = f$$

Then the solution of $AX = B$ is

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}B$$

inverse of a A is of the form

$$A^{-1} = \frac{1}{det(A)} \bullet A$$

to simplify the we let the system

$$2x + 3y + z = 2$$
$$4x + 3y + z = 3$$
$$2x + y + 3z = 2$$

the *augmented matrix* has a solution

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 2 \\ 4 & 3 & 1 & 3 \\ 2 & 1 & 3 & 1 \end{array}\right)$$

To get an upper(or right) triangular matrix. The Gauss elimination which gives us a triangular system

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 2 \\ 0 & -3 & -1 & -1 \\ 0 & 0 & 8 & 2 \end{array}\right)$$

To get a matrix in an echelon form note that the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 4 \\ 0 & 0 & 3 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

and the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 6 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 & 4 \\ 3 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Is not echelon form

and the solution by substituting back $(2/5,\ 3/5,\ 1/4)$. We can pursue with Jordan method

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2/5 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 1 & 1/4 \end{array}\right)$$

to find the solution as before. Jordan method gives a matrix called *row canonical form*. There is a relation between the solution of a linear system $AX = b$ an determinant of A:

Theorem:

$-det(A) = 0$ the system if *inconsistent* there is no solutions.
$-det(A) \neq 0$ the system if *consistent* , it may have a unique solution or an infinity of them. solutions.

to solve a linear system $AX = b$ we can use the *LU factorization*[12]. Let A be a matrix does $A = LU$ where L is *lower* triangular matrix and U the *upper* triangular matrix. Then

$$\begin{cases} AX = b \\ LUX = b \\ UX = Y \\ LY = b \end{cases}$$

[12] This decomposition was first used by Polish Tadeusz **Banchiewish**$(1882 - 1954)$ in 1938.

so to solve the original system $AX = b$ by solving two systems : $LY = b$ and $UX = Y$ by mere substitution.

Exercices

♦ Let $AX = \mathbf{0_n}$ and $det(A) \neq 0$. What is the solution of the equation?

♦ Find I_n for : $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

♦ Find I_n for : $\begin{pmatrix} 2 & 3 \\ 2 & 3 \\ 6 & 5 \end{pmatrix}$

♦ show that AA^T is always symmetric.

♦ show that : $A - A^T$ is skew.

♦Let A be a real symmetric matrix. Show that A is Hermitian.

♦ show that A^{-1} is unique.

♦ show that $(A^T)^{-1} = (A^{-1})^T$.

♦ give an example of : $AB \neq BA$.

♦ give an example of : $AB = \mathbf{0_{nn}}$.

♦ Show that $(AB)^{-1} = B^{-1}A^{-1}$.

♦ show that $det(A^{-1}) = \frac{1}{det(A)}$ What do you infer?

♦Let A be a regular symmetric matrix. Show that A^{-1} is symmetric.

♦ Show that : $det(A^T A) \geq 0$.

♦ find the a solution for the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \\ 2 & 2 & 3 & 3 \end{array}\right)$$

♦Find the eigenvalues and eigenvector of the following matrices:

– $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

– $\begin{pmatrix} a_{11} & a_{11} & a_{11} \\ 0 & a_{11} & a_{11} \\ 0 & 0 & a_{11} \end{pmatrix}$

♦ By two methods, show that the eigenvalues of $\begin{pmatrix} 1 & -i \\ i & 2 \end{pmatrix}$ are real.

♦ show by two methods that I_n is positive definite.

Vector spaces

In mathematics, a *space* is a set of elements with a certain structure in it. Many of the precedent results, can be incorporated in theory of vector spaces. A *vector space* ,from its name, is a space of vectors is of elements with a structure that is with linearity. The set $\mathbb{R}^n$ is a vector space whose vectors are $v = (a_1, \dots, a_n)$ that the following propties:

Definition

$-(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$
$-v + 0 = 0$
$-v + (-v) = 0$

This structure is called is a *group*. We have an evident propriety namely $v_1 + v_2 = v_2 + v_1$ then the group is *commutative*. We conclude that $(\mathbb{R}^n, +)$ is a commutative group. The following are also true:

-V+
-
-

We say technically that a set of points V (elements not necessarily numbers) is a K −espace vectorial is

Definition

- V is a commutative group[13].
$-\alpha \bullet (\beta \bullet v) = (\alpha\beta) \bullet v$
$-(\alpha + \beta) \bullet v = \alpha \bullet v + \beta \bullet v$
$-(v_1 + v_2) \bullet \alpha = \alpha \bullet v_1 + \beta \bullet v_2$
−for every $v \in V$ we have : $1 \bullet v = v$

[13] Condition can be formulated : $(v_1 + v_2)(\lambda +) = \lambda v_1 + v_2$.

So we have two sets, the set of $\alpha, \beta, \ldots$ which are *scalars* and the set V which are *vectors*. and the product by a map $K \times V \longrightarrow V$. The elements of a vector space are vectors , but not necessarily vectors of $\mathbb{R}^n$ but elements like function, matrices,… so a vector space is a general abstract structure.

Examples:

–$\mathbb{R}^n$ is a vector space by excellence. The elements of $\mathbb{R}^n$ are the vectors $(x_1, \ldots, x_n)$ of real numbers. One has to make a difference between the number 0 which is a scalar and the vector $\mathbf{0}$ which is $(0, \ldots, 0)$.
–the set of functions is a vector space ⏏

sub space

A subspace I ano the vector space in V.

Definition
a isa subspace of V if

-
-

Examples:

–The line $y = x$ is subspace of $\mathbb{R}^2$.

–is a subspace $\mathbf{0}$ is a subspace of $\boldsymbol{V}$⏏

Basis of a linear space

A linear space has a *basis* noted $(e_i)_{i=1}^n$ means that every $v \in V$ can be unqiely expressed in $\mathcal{B}$ in it

Theorem

Every vector space V has a basis.

The proof of this result uses the axiom of choice which is difficult to deal with here. We use also the *Zorn*[14] *lemma* that says : every has maximal. Its proof depends upon the *axiom of choice*, it is very used in the infinite process.

[14] Due to German mathematician Max **Zorn**$(1906 - 1993)$, who introduced it in 1935 when he was in the USA.

So a basis for V is linearly independent and spans V. The number of the elements in the basis $\mathcal{B}$ is the *dimension* of V. Any vector has a unit vector in its direction that can be of *direction cosines*

$$u_n = \frac{v}{\|v\|} = \frac{a_1}{\|v\|}e_1 + \frac{a_2}{\|v\|}e_2 + \cdots + \frac{a_n}{\|v\|}e_n = cos(\alpha_1)e_1 + cos(\alpha_2)e_2 + \cdots + cos(\alpha_n)e_n$$

And has a relation to unit vector.

Remarks:

– A basis can be not unique.

–If the basis is finite$(e_1, e_2, e_3, \dots, e_n)$ it is called a *Hamel*[15] *basis* , other if the basis $(e_1, e_2, e_3, \dots)$ is infinite, it is *Schauder*[16] *basis*. Vector spaces with infinite basis are used in functional analysis⏏

An independent $(e_1, e_2, e_3, \dots, e_n)$ is basis for $\mathbb{R}^n$. A collection of n vectors is a basis for $\mathbb{R}^n$ if the determinant is different from 0

$$det(e_1, e_2, e_3, \dots, e_n) \neq 0$$

A matrix can change according to the basis

Dimension of a vector space :

The dimension of vector space is the linearly independent vectors. It is the same is the of the basis.

Remarks:

– The dimension of subspace $\mathbf{0}$ is 0.

Orthonormal basis :

To any non-zero vector v, we can here is a vector defined by the expression $v_n = \frac{v}{\|v\|}$, it is called the *unit vector* it has the norm $\|v_n\| = 1$. Two *orthogonal* vectors v_1 and v_2 are vectors such that $v_1 \circ v_2 = 0$. *Orthonormal* vectors are orthogonal vectors with norm 1

Definition

An *orthonormal* basis is a basis all vector are of norm 1 and orthogonal two by two.

[15] Due to German mathematician Georg **Hamel**$(1877 - 1945)$.

[16] Due to Polish mathematician Juliusz **Schauder**$(1899 - 1943)$.

There is a method to transform a basis to orthogonal basis according to Gram–Schmidt method[17].

Rank of a linear system

We have that a linear system can either have a solution or not this depends also on then to the concept of *rank* of a linear system $AX = b$

Definition

The rank of a linear system is the number of linearly independent rows (or columns).

we will rank of a matrix A rather a system linear $AX = b$. that a matrix has a rank , the determinant is 0.

Changing the basis

Let V a linear space and $\mathcal{B}_1$, $\mathcal{B}_2$ two basis of V. we what is the matrix of from $\mathcal{B}_1$ to $\mathcal{B}_2$. it is simply the square matrix P such that : $\mathcal{B}_2 = P\mathcal{B}_1$. so if A is f and B is f in basis then the matrix to we have

$$A = PBP^{-1}$$

Exercices

♦ Let V a vector space. Show that
– $0v = \mathbf{0}$
– $\alpha \boldsymbol{v} = \mathbf{0}$
♦ Show that $\mathbf{0}$ is a subspace of any vector space.
♦Are the following subspaces of $\mathbb{R}^2$
– $(x, 0)$
– $(x, 1)$
– $(x, y - 1)$

♦Are the following subspaces of $\mathbb{R}^3$
– $(x, 0,0)$
– $(x, 0,0)$
– $(x, 0,0)$
♦ give a collection of vectors that can span $\mathbb{R}^3$.
♦ show that the representation of a vector in a basis is unique.
♦ show that $\{(1,2,3), (2,3,1)(-1,2,4)\}$ is a basis for $\mathbb{R}^3$.
♦ Is $\{(2, 1)(-2, -1)\}$ a basis for $\mathbb{R}^2$.

[17] Due to Danish Jørgen **Gram**$(1850 - 1916)$ and German Erhard **Schmidt**$(1876 - 1959)$.

Linear maps

A lot of linear algebra is about *linear maps*.

Definition

A map $f: V_1 \longrightarrow V_2$ is linear if $f(x+y) = f(x+f(y)$ and $f(ax) = af(x)$

we both by writing $f(ax+by) = af(x) + bf(y)$. if $f: E \longrightarrow F$ if EF are not vector spce we cannot talk about linear map. A linear map is an example of general concept called *homomorphism* between two groups.

We are interested in maps between vector spaces.

Remarks:

– A Linear map $f: V_1 \longrightarrow V_2$ can presented by an infinity of square matrices from vector space to itself is called *endomorphism*, any endomorphism can presented by an infinity of square matrices ⏏

Examples:

–The numerical map $y = ax$ is linear, $y = ax + b$ is *affine*.

–The two[18] maps $\frac{d}{dx}$ and $\int$ are linear

–the map $f(x_1, \dots, x_1) = a_1x_1 + a_1x_2 + \cdots + a_nx_n$ is a linear map⏏

***The image and kernel*:**

Let $f: V_1 \longrightarrow V_2$ linear so the image of is y such that $f(x) = y$.

[18] The two are called operators in functional analysis.

The kernel: the is the x $f(x) = 0$ a ker is homomorphism between two groups. So we have for linear $f(0) = 0$

(f is a morphism , $f(e_1) = e_2$). The element 0 is always in $Ker(f)$ and element of , and $ax + b$ is not linear.
We have theorem

Theorem

Ker is a subspace of V_1 and Im is a subspace of V_2

H

A linear $f : V_1 \longrightarrow V_2$. We define

Theorem

$$Dim(V_1) = dimKer + dim(\ Im\ )$$

Linear maps and matrices:

To every map : $\mathbb{R}^m \longrightarrow \mathbb{R}^n$, i.e. between two vector spaces, there corresponds a matrix of n lines and m columns. A squared matrix is by $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$
we know that the basis is unique so a matrix of passage from $(e_i)_{i=1}^n$ to $(e_i)_{i=1}^n$. A known result in linear algebra says :

Theorem

To every matrix corresponds a linear map and vice versa[19]. A matrix of order nm is a function from a linear space of dimension m to another linear space of dimension n.

to a linear map there are infinitely many matrices, because of the of the basis

Exercices

- Show that if $(Ker(f) = 0_E)$ then f is injective.
- Show that $Ker(f)$ is a subspace of E.
- show that $Im(f)$ is a subspace of F.
- show that if f is linear then E and F are vector spaces. Is the inverse true ?

[19] When the space E is an infinite dimensional space, a linear map may be not presented by a matrix.

♦ Let f be a linear map. Show that $f(0) = 0$.

♦ Say if the following f are linear maps.

– $f(x) = x + 1$

– $f(x, y) = xy + x$

– $f(x, y) = x + y + 1$

– $f(x, y) = x + y.$

♦ give the correspondent maps to following matrices.

– $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

– $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

– $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

– $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

♦ Give the correspondent matrices to following linear maps.

– $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ y - 2y \end{pmatrix}$

– $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y - z \\ y + y + z \end{pmatrix}$

– $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

– $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

Spectrum and eigenvectors

The motivation for eigenvalue is as follow : we know that Av gives another vector u so A transforms a vector v to another vector it can be $v = u$. The u can be same direction as v. We look that do not change the direction of the vector v but only λv

$$Av = \lambda v$$

so we look for the *Eigen* vector s to find the *proper* or own values. To find the eigenvector of A is of the form $Av = \lambda v$ gives $Av - \lambda Iv = 0$ or $(A - \lambda I)v = 0$ so either or is $\mathbf{0_n}$ or $v = (0, \dots, 0)$ but . the solution is that should be singular which means

$$det(A - \lambda I) = 0$$

It is called the *characteristic equation*. So by the solution of the equation we find *eigenvalue*[20] of A. The set of the eigenvalues a matrix A is called the *specter* of A. To find the eigenvector corresponding to eigenvalue λ we solve the equation

$$(A - \lambda I)v = 0$$

Remarks:

– If a matrix A has an eigenvalue $\lambda = 0$ then is singular and its determinant is 0 ⏏

Examples:

– Find the eigenvalues of $\begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$. $det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 3 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)$, the equation gives us two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$ which constitute the $spect(A) = \{1,4\}$. To an eigenvalue correspond an eigenvector : $Av = \lambda v$.for the first $\lambda_1 = 1$ we solve the $(A - I)v = 0$

[20] From German *Eigen* means "*own*" , "*proper*" .

⏏

Theorem

If a matrix A has distinct eigenvalues then the corresponding eigenvectors are linearly independent.

The *spectral radius* of a matrix A is

$$\rho(A) = max\{|\lambda_i|\}$$

It is evident that the eigenvalues the characteristic equation. Is the Caylay—Hamilton[21] theorem or properly theorem that the matrix A itself the characteristic equation.

Cayley—Hamilton Theorem

Let A be a matrix of order n then the characteristic equation.

The eigenvalues o f a symmetric matrix are real.

Theorem

A and A^T have the same eigenvalues.

When the eigenvalues of a matrix are then what is a positive definite matrix. The eigenvalues of a symmetric are orthogonal and eigenvalues are real.

Theorem

Let A be a matrix of order n then

$$det(A) = \prod_{i=1}^{i=n} \lambda_i \quad , \quad Tr(A) = \sum_{i=1}^{i=n} \lambda_i$$

Example:

—the eigenvalues of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ comes as the solution of the equation

$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 3 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4$, so we have

[21] It was by Arthur **Caylay**$(1821 - 1895)$ and William **Hamilton**$(1805 - 1865)$ for specila cases. But the general case was proved by Ferdinand **Frobenius**$(1849 - 1917)$. It should be called the *Frobenius theorem*.

$$\begin{pmatrix}1 & 0\\ 3 & 4\end{pmatrix}^2 - 5\begin{pmatrix}1 & 0\\ 3 & 4\end{pmatrix} + 4\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

⏏

Matrix definiteness:

A matrix can be by the quadratic form : $x^T Ax$, where x is the n −vector and x^T its transpose.

Proposition

A matrix A is *positive definite* if : $x^T Ax \geq 0$

Example:

– Let $A = \begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}$ then $(x_1 \ x_2)\begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = (x_1 + 2x_2 \quad x_1 + x_2)\begin{pmatrix}x_1\\ x_2\end{pmatrix} = 2{x_1}^2 + 3{x_2}^2$ which is evidently > 0 , then the matrix A is positive definite.

⏏

Proposition

A matrix A is *negative definite* if : $x^T Ax \leq 0$

Example:

– Let $A = \begin{pmatrix} & 1\\ 2 & 1\end{pmatrix}$ then $(x_1 \ x_2)\begin{pmatrix}1 & \\ & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = (x_1 + 2x_2 \quad x_1 + x_2)\begin{pmatrix}x_1\\ x_2\end{pmatrix} = 2{x_1}^2 + 3{x_2}^2$ which is evidently > 0 then the matrix A is definite .

⏏

We by using eigenvalues

Let A a squared matrix and

Proposition:

If $\lambda_i > 0$ is a positive definite matrix
If $\lambda_i \geq 0$ is a positive semi-definite matrix

If $\lambda_i \leq 0$ is a negative semidefinite matrix
If $\lambda_i < 0$ is a negative semidefinite matrix

Remarks:

– A matrix can be neither ⏏

Minimal polynomial:

The *minimal polynomial* is a polynomial with the least degree with $a_n = 1$ that verified by the eigenvalues.

Proposition

If the eigenvalues of a matrix A are different then the minimal polynomial and the characteristic polynomial are the same.

Norm of a matrix :

we can associate as we did to a vector, a norm to a matrix. Let A be a matrix, the following are possible norms of A used :

$$\|A\|_1 = \sum |a_{ij}|$$
$$\|A\|_2 = \sum |a_{ij}|$$

We have

$$\|AB\| \leq \|A\|\|B\|$$

$$\rho(A) \leq \|A\|$$

Where ρ the spectral radius.

Exercices

♦Show that if α is an eigenvalue for A then $\frac{1}{\alpha}$ is eigenvalue for A^{-1} .

♦Let $A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 1 \\ 5 & 1 & -1 \end{pmatrix}$. Find its spectral radius.

Reduction

In relation with the solution of linear systems is the *reduction* of matrices. Reduction is usually simplify the solution of a linear system, so this means to transform a matrix A in a suitable form to calculate. We look for a simpler matrix . the simplest matrix A is the *diagonal* matrix of the form

$$D = \begin{pmatrix} \beta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \beta_n \end{pmatrix}$$

We know a matrix A can change according to the basis

Theorem

A diagonal matrix to find a basis diagonalizable.

Where all zeros except the diagonal or $a_{ij} = \beta_i \delta_{ij}$. The unit matrix I_n is a diagonal matrix. that a form a *diagonalizable*. have that is triangular or echelon which gives the solution of the system.

Definition

A squared matrix is *diagonalizable* iff it is similar to a diagonal matrix : $PAP^{-1} = D$

or the same $A = PDP^{-1}$, we made a decomposition of A into three matrices. For example it is easy to find the A^n , it is just

$$A^n = PDP^{-1}PDP^{-1} \ldots PDP^{-1} = PD^nP^{-1}$$

There are matrices that are not diagonalizable, the cause is that:

-some have multiplicity of eigenvalues which are different from the multiplicity of the eigenvectors.

Diagonalization of matrices:

We have learned that a linear map can differently represented by a matrix according to the chosen basis. So what is the simplest matrix corresponding to f? So when $P^{-1}AP$ is diagnosable.

Theorem

A matrix is diagonalizable iff there is a basis composed of eigenvectors.

The idea is to find a basis which give a reduced matrix . so find a basis such that is triangular.

Theorem

If the eigenvalues of A are different, then A is diagonisable.

then $det(A) = \prod_{i=1}^{n} a_{ii}$

Proposition:

Every Hermitian matrix is diagonalizable.

Example:

–the⏏

If A is *similar* to D then $PAP^{-1} = D$. The matrix is formed of the eigenvalues of A.

Theorem

Two similar matrices have the same eigenvalues.

Let A be a square matrix then:

Theorem

-if A is normal, there is a unitary U such that $U^{-1}AU$ is diagonal.
- if A is symmetric , there is orthogonal matrix O such that $O^{-1}AO$ is diagonal.

Triangulation of matrices:

in other cases we cannot find a basis diagonal, then at least triangular. A *triangular* matrix are two is of the form can be

$$\begin{pmatrix} a_{11} & a_{11} & a_{11} \\ 0 & a_{11} & a_{11} \\ 0 & 0 & a_{11} \end{pmatrix}$$

A *triangular* matrix is of the form

$$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{11} & a_{11} & 0 \\ a_{11} & a_{11} & a_{11} \end{pmatrix}$$

Let A be a square matrix then:

Theorem

-there is a unitary matrix U such that $U^{-1}AU$ is triangular.

Jordan form:

There are matrices that are not diagonalizable, some have multiplicity of eigenvalues which are different from the multiplicity of the eigenvectors. So Jordan theorem gives $PAP^{-1} = D$ gives $PDP^{-1} = A$ then a *Jordan normal form* D we name J.

$$A = PJP^{-1}$$

$$\begin{pmatrix} J00 \\ 0J0 \\ 00J \end{pmatrix}$$

S oa result

Theorem

For every matrix A there is a Jordan form : $J = P^{-1}AP$

So the theorem says that every matrix A is similar to a Jordan normal form.

Example:

–the

Exercices

♦ Calculate $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n$.

♦ Let A and B be similar matrices. Show that $det(A) = det(B)$